

Harnack Inequalities and Applications for Stochastic Differential Equations Driven by Fractional Brownian Motion

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Abstract. In the paper, Harnack inequalities are established for stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. As applications, strong Feller property, log-Harnack inequality and entropy-cost inequality are given.

Mathematics Subject Classifications (2000): Primary 60H15

Key words and phrases: Harnack inequality, stochastic differential equation, fractional Brownian motion.

1 Introduction

Under a curvature condition, Wang [21] introduced dimensional-free Harnack inequality for diffusions on Riemannian manifold. This type of inequality has been studied extensively, see, for example, Aida and Kawabi [2, 3] for infinite dimensional diffusion processes; Wang [24] for stochastic generalized porous media equations; Röckner and Wang [17] for generalizes Mehler semigroup; [1] for stochastic functional differential equation; Ouyang [20] for Ornstein-Uhlenbeck processes and multivalued stochastic evolution equations etc.

Harnack inequality has various applications, see, for instance, [8, 17, 18, 22, 23] for strong Feller property and contractivity properties; [2, 3] for short times behaviors of infinite dimensional diffusions; [8, 11] for heat kernel estimates and entropy-cost inequalities. [2, 12, 17, 21] established Harnack inequalities using the method of derivative formula. In order to obtain Harnack inequality on manifolds with unbounded below curvatures, [5] introduced the approach of coupling and Girsanov transformations. In the paper, we will use the above two methods to establish Harnack inequalities for stochastic differential equations driven by fractional Brownian motion.

Solutions of the stochastic differential equations driven by fractional Brownian motion have been studied intensively in recent years, for example see [13, 15] using the pathwise approach; see [9] using the tools of rough path analysis introduced in [13]. We prove Harnack inequality for stochastic differential equations driven by fractional Brownian motion with Hurst parameter

$H < \frac{1}{2}$. As applications of the Harnack inequality, the strong Feller property and the log-Harnack inequality are derived. We also get the entropy-cost inequality with respect to the Euclidian distance.

The paper is organized as follows. In section 2, we give some preliminaries on fractional Brownian motion. Section 3 prove the Harnack inequality by using the approach of coupling and Girsanov transformations, and present their applications. In section 4, we are devoted to establish derivative formula and give the corresponding Harnack inequality.

Harnack inequality

2 Preliminaries

Let $B^H = \{B_t^H, t \in [0, T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ defined on the probability space $\Omega, \mathcal{F}, \mathbb{P}$, i.e., B^H is a centered Gauss process with the covariance function

$$R_H(t, s) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

In particular, if $H = \frac{1}{2}$, B is a Brownian motion. It is well known that if $H \neq \frac{1}{2}$, B^H does not have independent increments and has α -order Hölder continuous path for all $\alpha \in (0, H)$.

For each $t \in [0, T]$, we denote by \mathcal{F}_t the σ -algebra generated by the random variables $\{B_s^H : s \in [0, T]\}$ and the \mathbb{P} -null sets.

We denote by \mathcal{E} the set of step functions on $[0, T]$. Let \mathcal{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]} \rangle R_H(t, s) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

The mapping $I_{[0,t]} \mapsto B_t^H$ can be extended to an isometry between \mathcal{H} and the Gauss space \mathcal{H}_1 associated with B^H . Denote this isometry by $\phi \mapsto B^H(\phi)$. For more details, one can see [16]. On the other hand, from [10], we know the covariance kernel $R_H(t, s)$ can be written as

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr,$$

where K_H is a square integrable kernel given by

$$K_H(t, s) = \Gamma(H + \frac{1}{2})^{-1} (t - e)^{H - \frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}),$$

in which $F(\cdot, \cdot, \cdot, \cdot)$ is the Gauss hypergeometric function.

Define the linear operator $K_H^* \mathcal{E} \rightarrow L^2[0, T]$ as follows

$$(K_H^* \phi)(s) = K_H(t, s) \phi(s) + \int_s^T (\phi(r) - \phi(s)) \frac{\partial K_H}{\partial r}(r, s) dr.$$

By [4], we know that, for all $\phi, \psi \in \mathcal{E}$, $\langle K_H^* \phi, K_H^* \psi \rangle_L^2[0, T] = \langle \phi, \psi \rangle$ holds. From B.L.T. theorem, K_H^* can be extended to an isometry between \mathcal{H} and $L^2[0, T]$. Therefore, according to [4], the

process $\{W_t = B((K_H^*)^{-1}(I_{[0,t]})), t \in [0, T]\}$ is a Wiener process, and B^H has the following integral representation

$$B_t^H = \int_0^t K_H(t, s) dW_s.$$

By [10], the operator $K_H : L^2[0, T] \rightarrow I_{0+}^{H+\frac{1}{2}}(L^2[0, T])$ associated with the square integrable kernel $K_H(\cdot, \cdot)$ is defined as follows

$$(K_H f)(t) := \int_0^t K_H(t, s) f(s) ds, \quad f \in L^2[0, T].$$

where $I_{0+}^{H+\frac{1}{2}}$ is the α -order left fractional Riemannian-Liouville integral operator on $0, T$, one can see [19]. It is an isomorphism and for each $f \in L^2[0, T]$,

$$\begin{aligned} (K_H f)(s) &= I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f, \quad H \leq \frac{1}{2}, \\ (K_H f)(s) &= I_{0+}^1 s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f, \quad H \geq \frac{1}{2}. \end{aligned}$$

As a consequence, for every $h \in I_{0+}^{H+\frac{1}{2}}(L^2[0, T])$, the inverse operator K_H^{-1} is of the following form

$$\begin{aligned} (K_H^{-1} h)(s) &= s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h', \quad H > \frac{1}{2}, \\ (K_H^{-1} h)(s) &= s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} h, \quad H < \frac{1}{2}, \end{aligned}$$

where $D_{0+}^{H-\frac{1}{2}}(D_{0+}^{\frac{1}{2}-H})$ is $H - \frac{1}{2}(\frac{1}{2} - H)$ -order left-sided Riemannian-Liouville derivative, one also can see [19].

In particular, if h is absolutely continuous, we have

$$(K_H^{-1} h)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} h', \quad H < \frac{1}{2}.$$

In [14], D.Nualart and Y.Ouknine discussed the following stochastic differential equations driven by fractional Brownian motion on \mathbb{R} ,

$$dX_t = b(t, X_t)dt + dB_t^H, \quad X_0 = x. \quad (2.1)$$

They proved the existence and uniqueness of a strong solution for (2.1) when $b(t, x)$ is a Borel function with linear growth in x in case $H \leq \frac{1}{2}$.

The aim of the paper is to consider the Harnack inequality for the equation (2.1) in case $H < \frac{1}{2}$. We define $P_t f(x) := \mathbb{E}f(X_t^x)$, $t \in [0, T]$, $f \in \mathcal{B}_b(\mathbb{R})$, where X_t^x is the solution to the equation (2.1) and $\mathcal{B}_b(\mathbb{R})$ denotes the set of all bounded measurable functions on \mathbb{R} .

3 Main results and proofs

Let us start with the following hypothesis (H1):

- (i) $|b(t, x) - b(t, y)| \leq K|x - y|$, $\forall x, y \in \mathbb{R}, t \in [0, T]$, where $K > 0$ is a constant;

(ii) The mapping $t \mapsto b(t, 0)$ is bounded on $[0, T]$.

It is clear that under (H1), the equation (2.1) has a unique solution. Furthermore, we can give the Harnack inequality for the equation (2.1) as follows.

Theorem 3.1 *If (H1) holds, then for any nonnegative $f \in \mathcal{B}_b(\mathbb{R})$ and $t > 0, x, y \in \mathbb{R}$,*

$$(P_T f(y))^p \leq P_T f^p(x) \exp\left[\frac{p}{p-1} C(T, K, H) |x - y|^2\right],$$

where $C(T, K, H) = \left(\frac{B(\frac{3}{2}-H, \frac{1}{2}-H)}{\Gamma(\frac{1}{2}-H)}\right)^2 \frac{T^{2-2H}}{K^{-2}(1-e^{-2KT})^2(1-H)}$.

Proof. The proof will be divided into three steps.

Step 1. Consider the following coupled stochastic differential equation

$$dY_t = b(t, Y_t)dt + dB_t^H + u_t dt, \quad Y_0 = y, \quad (3.1)$$

$$dX_t = b(t, X_t)dt + dB_t^H, \quad X_0 = x, \quad (3.2)$$

where the drift term u_t of the equation (3.1) is of the following form

$$\eta_t \cdot \frac{X_t - Y_t}{|X_t - Y_t|} I_{\{t < \tau\}},$$

τ is the coupling time of X_t and Y_t defined by

$$\tau = \inf\{t \geq 0 : X_t = Y_t\},$$

and η_t is a deterministic function on $[0, \infty)$ specified later such that the force u_t can make the two processes X and Y move together before time T .

It is obvious that the assumption (H1) implies $|b(t, x)| \leq C(1 + |x|)$, then, according to [14, Theorem 8], the equation (3.1) has a unique solution.

Note that $d(X_t - Y_t) = (b(t, X_t) - b(t, Y_t))dt - u_t dt$, thus applying the Tanaka formula to $|X_t - Y_t|$, we have for $t < \tau$

$$\begin{aligned} d|X_t - Y_t| &= \text{sgn}(X_t - Y_t) d(X_t - Y_t) \\ &= \text{sgn}(X_t - Y_t) (b(t, X_t) - b(t, Y_t)) dt - \eta_t dt. \end{aligned}$$

By (H1), for all $t < \tau$ we get

$$d|X_t - Y_t| \leq (K|X_t - Y_t| - \eta_t) dt.$$

This implies that

$$e^{-K(T \wedge \tau)} |X_T - Y_T| \leq |x - y| + \int_0^T e^{-Kt} \eta_t dt. \quad (3.3)$$

Choosing

$$\eta_t = \frac{e^{-Kt}}{\int_0^T e^{-2Kt} dt} \cdot |x - y|, \quad t \geq 0.$$

We conclude that $\tau \leq T$ and $X_T = Y_T, a.s.$ Otherwise, if $\tau > T$, by (3.3) we get $X_T = Y_T$. But this contradicts with the assumption that $\tau > T$.

Step 2. Let $\tilde{B}_t^H = \int_0^t u_s ds + B_t^H$, $\forall t \in [0, T]$. By simple calculus, we know that $\int_0^T u_t^2 dt < \infty$. Hence, $\int_0^\cdot u_r dr \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$. According to integral representation of fractional Brownian motion and the definition of the operator K_H , we deduce

$$\begin{aligned}\tilde{B}_t^H &= \int_0^t u_s ds + \int_0^t K_H(t, s) dW_s \\ &= \int_0^t K_H(t, s) \left[(K_H^{-1} \int_0^\cdot u_r dr)(s) ds + dW_s \right] \\ &=: \int_0^t K_H(t, s) d\tilde{W}_s.\end{aligned}$$

Now, let

$$R_T = \exp \left[- \int_0^T \left(K_H^{-1} \int_0^\cdot u_r dr \right) (s) dW_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot u_r dr \right)^2 (s) ds \right].$$

Next we want to show $(\tilde{B}_t^H)_{0 \leq t \leq T}$ is an $\mathcal{F}_t^{B^H}$ -fractional Brownian motion with Hurst parameter H under the new probability $R_T P$. Due to [14, Theorem 2], it suffices to show that $\mathbb{E} R_T = 1$. Since $\int_0^\cdot u_r dr$ is absolutely continuous, then

$$(K_H^{-1} \int_0^\cdot u_r dr)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} u_s.$$

Hence, we have

$$\begin{aligned}\left| (K_H^{-1} \int_0^\cdot u_r dr)(s) \right| &= \left| \frac{1}{\Gamma(\frac{1}{2}-H)} s^{H-\frac{1}{2}} \int_0^s r^{\frac{1}{2}-H} u_r (s-r)^{-H-\frac{1}{2}} dr \right| \\ &\leq \frac{1}{\Gamma(\frac{1}{2}-H)} s^{H-\frac{1}{2}} \int_0^s |\eta_r| r^{\frac{1}{2}-H} (s-r)^{-H-\frac{1}{2}} dr \\ &\leq \frac{1}{\Gamma(\frac{1}{2}-H)} \frac{|x-y|}{(2K)^{-1}(1-e^{-2KT})} s^{H-\frac{1}{2}} \int_0^s r^{\frac{1}{2}-H} (s-r)^{-H-\frac{1}{2}} dr \\ &= \frac{B(\frac{3}{2}-H, \frac{1}{2}-H)}{\Gamma(\frac{1}{2}-H)} \frac{|x-y|}{(2K)^{-1}(1-e^{-2KT})} s^{\frac{1}{2}-H}.\end{aligned}$$

As a consequence, we get

$$\mathbb{E} \exp \left[\frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot u_r dr \right)^2 (s) ds \right] \leq \exp[C(T, K, H)|x-y|^2], \quad (3.4)$$

where $C(T, K, H) = \left(\frac{B(\frac{3}{2}-H, \frac{1}{2}-H)}{\Gamma(\frac{1}{2}-H)} \right)^2 \frac{T^{2-2H}}{K^{-2}(1-e^{-2KT})^2(1-H)}$. Using the Novikov criterion, we have $\mathbb{E} R_T = 1$.

Step 3. From step 2, we can rewrite (3.1) in the following form

$$dY_t = b(t, Y_t) dt + d\tilde{B}_t^H, \quad Y_0 = y,$$

where $(\tilde{B}_t^H)_{0 \leq t \leq T}$ is an $\mathcal{F}_t^{B^H}$ -fractional Brownian motion with Hurst parameter H under the new probability $R_T P$. By the uniqueness of the solution and $X_T = Y_T, a.s.$, we have

$$P_T f(y) = \mathbb{E} f(X_T^y) = \mathbb{E} R_T f(Y_T^y) = \mathbb{E} R_T f(X_T^x). \quad (3.5)$$

Applying the Hölder inequality to (3.5), we obtain

$$(P_T f(y))^p \leq P_T f^p(x) \cdot (\mathbb{E} R_T^{\frac{p}{p-1}})^{p-1}. \quad (3.6)$$

Now we will estimate moments of R_T .

Denote $\alpha = \frac{p}{p-1}$ and $M_T = -\int_0^T (K_H^{-1} \int_0^\cdot u_r dr) (s) dW_s$. Since $(R_t)_{0 \leq t \leq T}$ is a \mathbb{P} martingale, by (3.4) we have

$$\begin{aligned} \mathbb{E} R_T^\alpha &= \mathbb{E} \exp[\alpha M_T - \frac{1}{2} \alpha \langle M \rangle_T] \\ &= \mathbb{E} \exp[\alpha M_T - \frac{1}{2} \alpha^2 \langle M \rangle_T + \frac{1}{2} \alpha(\alpha-1) \langle M \rangle_T] \\ &\leq \exp[\alpha(\alpha-1) C(T, K, H) |x-y|^2]. \end{aligned} \quad (3.7)$$

Substituting (3.7) into (3.6), we get the desired result.

Remark 3.2 In the proof of Theorem 3.1, the choice of u_t is not unique. For instance, we can take another as follows

$$u_t = \eta_t \cdot \frac{X_t - Y_t}{|X_t - Y_t|} I_{\{t < \tau\}}, \quad \eta_t = \frac{1}{\int_0^T e^{-Kt} dt} \cdot |x - y|.$$

Correspondingly, the result of Theorem 3.1 is of the following form

$$(P_T f(y))^p \leq P_T f^p(x) \exp[\frac{p}{p-1} \tilde{C}(T, K, H) |x-y|^2],$$

where $\tilde{C}(T, K, H) = \left(\frac{B(\frac{3}{2}-H, \frac{1}{2}-H)}{\Gamma(\frac{1}{2}-H)} \right)^2 \frac{T^{2-2H}}{4K^{-2}(1-e^{-KT})^2(1-H)}.$

As applications of Theorem 3.1, we prove the following results on strong Feller property for P_T and log-Harnack inequality.

Proposition 3.3 Assume (H1). Then P_T is strong feller and the following estimate holds

$$|P_T f(x) - P_T f(y)| \leq \|f\|_\infty [2C(T, K, H)]^{\frac{1}{2}} |x-y| \exp[C(T, K, H) |x-y|^2],$$

for every $T > 0, x, y \in \mathbb{R}$ and $f \in \mathcal{B}_b(\mathbb{R})$.

Proof. It follows from the proof of Theorem 3.1 that, for each $f \in \mathcal{B}_b(\mathbb{R})$,

$$|P_T f(x) - P_T f(y)| = |\mathbb{E} f(X_T^x) - \mathbb{E} R_T f(X_T^x)| \leq \|f\|_\infty \mathbb{E} |1 - R_T|. \quad (3.8)$$

Next we will estimate the term $\mathbb{E} |1 - R_T|$.

Firstly, we have

$$(\mathbb{E} |1 - R_T|)^2 \leq \mathbb{E} |1 - R_T|^2 = \mathbb{E} R_T^2 - 1. \quad (3.9)$$

Taking $\alpha = 2$ in (3.7), we have

$$\mathbb{E} R_T^2 \leq \exp[2C(T, K, H) |x-y|^2]. \quad (3.10)$$

Combining (3.9) with (3.10), we get

$$(\mathbb{E} |1 - R_T|)^2 \leq 2C(T, K, H) |x-y|^2 \exp[2C(T, K, H) |x-y|^2], \quad (3.11)$$

where we use the elementary inequality $e^x - 1 \leq x e^x$, $\forall x \geq 0$.

Substituting (3.11) into (3.8), we can deduce the desired result.

Corollary 3.4 *Let (H1) hold, then*

$$P_T(\log f)(x) \leq \log P_T f(y) + C(T, K, H)|x - y|^2,$$

$$\forall x, y \in \mathbb{R}, t > 0, f \geq 1, f \in \mathcal{B}_b(\mathbb{R}).$$

That is, log-Harnack inequality holds.

In fact, since \mathbb{R} is a length space, then, by [21, Proposition 2.2], we know the result holds.

To state further application of Theorem 3.1, let us introduce another assumption and some notations.

(H2): let μ be a probability measure on \mathbb{R} such that for some $\tilde{K} > 0$,

$$\mu(P_T f) \leq \tilde{K} \mu(f), \forall f \in \mathcal{B}_b^+(\mathbb{R}).$$

Note that if μ is P_T -invariant, then (H2) holds.

Remark 3.5 *The measures μ satisfying (H2) always exist. For instance,*

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} P_T^n(x, \cdot), \quad \forall x \in \mathbb{R},$$

where $(P_T^n(x, \cdot))_{n \geq 1}$ is defined recursively as follows

$$P_T(x, A) := P_T I_A(x), \quad P_T^n(x, A) := \int_{\mathbb{R}} P_T^{n-1}(x, dy) P_T(y, A), \quad n \geq 2.$$

Let $\mathcal{C}(\mu, \nu)$ denote the set of all couplings of μ and ν , where μ and ν are two given probability on \mathbb{R} , and $W_2(\mu, \nu)$ be the L^2 -Wasserstein distance between them with respect to the Euclidian distance, i.e.

$$W_2^2(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y|^2 \pi(dx, dy).$$

Corollary 3.6 *Assume that (H1) holds and μ satisfies (H2) ($\tilde{K} = 1$). Then the following entropy-cost inequality holds for each $T > 0$ and $f \in \mathcal{B}_b^+(\mathbb{R})$ with $\mu(f) = 1$,*

$$\mu(P_T^* f \log P_T^* f) \leq C(T, K, H) W_2^2(\mu, f\mu),$$

where P_T^* is the adjoint operator of P_T in $L^2(\mu)$.

Proof. By Corollary 3.4 for $P_T^* f$, we have

$$P_T(\log P_T^* f)(x) \leq \log P_T(P_T^* f)(y) + C(T, K, H)|x - y|^2, \quad \forall x, y \in \mathbb{R}. \quad (3.12)$$

Integrating both sides of (3.12) with respect to $\pi \in \mathcal{C}(\mu, f\mu)$, we get

$$\mu(P_T^* f \log P_T^* f) \leq \mu(\log P_T(P_T^* f)) + C(T, K, H) \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y|^2 \pi(dx, dy).$$

Note that, the Jensen inequality and the hypotheses imply

$$\mu(\log P_T(P_T^* f)) \leq \log \mu(P_T(P_T^* f)) \leq \log \mu(P_T^* f) = \log \mu(f P_T 1) = \log \mu(f) = 0.$$

So, we get

$$\mu(P_T^* f \log P_T^* f) \leq C(T, K, H) \inf_{\pi \in \mathcal{C}(\mu, f\mu)} \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y|^2 \pi(dx, dy).$$

The proof is complete.

4 Derivative formula

In this part, we begin with the following hypothesis (H3):

- (i) $\partial_2 b(t, x) \leq \overline{K}$, $\forall x \in \mathbb{R}, t \in [0, T]$, where $\overline{K} > 0$ is a constant, where $\partial_2 b(t, x)$ denotes the derivative for the second variable;
- (ii) The mapping $t \mapsto b(t, 0)$ is bounded on $[0, T]$.

The aim is to establish a Bismut type derivative formula for P_T which will imply the Harnack inequality. For $f \in \mathcal{B}_b(\mathbb{R})$, $x, y \in \mathbb{R}, T > 0$, we will consider

$$D_y P_T f(x) := \lim_{\epsilon \rightarrow 0} \frac{P_T f(x + \epsilon y) - P_T f(x)}{\epsilon}.$$

Theorem 4.1 (Derivative formula) Assume (H3). Then, for each $T > 0, f \in \mathcal{B}_b(\mathbb{R}), x, y \in \mathbb{R}$, $D_y P_T f(x)$ exists and satisfies

$$D_y P_T f(x) = \mathbb{E} f(X_T^x) N_T,$$

$$\text{where } N_T = \frac{1}{\Gamma(\frac{1}{2}-H)T} \int_0^T s^{H-\frac{1}{2}} \left[\int_0^s \frac{r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} (1 + \partial_2 b(r, X_r)(T-r)) dr \right] y dW_s.$$

Proof. As above, X_t^x solves the equation (2.1). For any $\epsilon > 0$ and $y \in \mathbb{R}$, we introduce the following equation

$$dX_t^\epsilon = b(t, X_t)dt + dB_t^H - \frac{\epsilon}{T}y, \quad X_0^\epsilon = x + \epsilon y. \quad (4.1)$$

By (H3), we easily know that the above equation has a unique solution. Combining (2.1) with (4.1), we deduce that $X_t^\epsilon - X_t = \frac{T-t}{T}\epsilon y$, $\forall t \in [0, T]$, in particular, $X_T^\epsilon = X_T$. Let $\eta_t = b(t, X_t) - b(t, X_t^\epsilon) - \frac{\epsilon}{T}y$, $\forall t \in [0, T]$, then we can rewrite (4.1) in the form:

$$dX_t^\epsilon = b(t, X_t^\epsilon)dt + d\overline{B}_t^H,$$

where $\overline{B}_t^H = B_t^H + \int_0^t \eta_s ds$. Note that

$$|\eta_t| \leq \overline{K}|X_t - X_t^\epsilon| + \frac{\epsilon y}{T} = \overline{K} \frac{T-t}{T} \epsilon y + \frac{\epsilon y}{T},$$

so, we have $\int_0^T \eta_t^2 dt \leq \infty$, and moreover, $\int_0^\cdot \eta_r dr \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$. Due to the integral representation of fractional Brownian motion and the definition of the operator K_H , we get

$$\overline{B}_t^H = \int_0^t K_H(t, s) d\overline{W}_s,$$

where $\overline{W}_t = W_t + \int_0^t (K_H^{-1} \int_0^\cdot \eta_r dr)(s) ds$. Now, let

$$R_\epsilon = \exp \left[- \int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right) (s) dW_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right)^2 (s) ds \right]$$

Now we will prove that $(\overline{B}_t^H)_{0 \leq t \leq T}$ is an $\mathcal{F}_t^{B^H}$ -fractional Brownian motion with Hurst parameter H under the new probability $R_\epsilon P$, according to [14, Theorem 2], it only needs to show $\mathbb{E}R_\epsilon = 1$. Similar to step 2 of theorem 3.1, we get

$$\left| (K_H^{-1} \int_0^\cdot \eta_r dr)(s) \right| \leq \frac{B(\frac{3}{2} - H, \frac{1}{2} - H)}{\Gamma(\frac{1}{2} - H)} \epsilon y (\overline{K} + \frac{1}{T}) s^{\frac{1}{2} - H}. \quad (4.2)$$

Hence, it follows that

$$\mathbb{E} \exp \left[\frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right)^2 (s) ds \right] < \infty.$$

By the Novikov criterion, $\mathbb{E}R_T = 1$ holds.

Hence, in view of the uniqueness of the solution and $X_T^\epsilon = X_T$, we have

$$P_T f(x + \epsilon y) = \mathbb{E} R_\epsilon f(X_T^x).$$

With the help of the dominated convergence theorem due to (4.2), we deduce that

$$\begin{aligned} D_y P_T f(x) &:= \lim_{\epsilon \rightarrow 0} \frac{P_T f(x + \epsilon y) - P_T f(x)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \left[\mathbb{E} f(X_T^x) \frac{R_\epsilon - 1}{\epsilon} \right] \\ &= \mathbb{E} \left[f(X_T^x) \lim_{\epsilon \rightarrow 0} \frac{R_\epsilon - 1}{\epsilon} \right]. \end{aligned}$$

Let $\widetilde{M}_T =: - \int_0^T (K_H^{-1} \int_0^\cdot \eta_r dr)(s) dW_s$. Thanks to (4.2), we get

$$\langle \widetilde{M} \rangle_T = \int_0^T |(K_H^{-1} \int_0^\cdot \eta_r dr)(s)|^2 ds \leq C \epsilon^2,$$

where C is a positive constant. Therefore, we deduce

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{R_\epsilon - 1}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \frac{\exp[\widetilde{M}_T - \frac{1}{2} \langle \widetilde{M} \rangle_T] - 1}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\widetilde{M}_T - \frac{1}{2} \langle \widetilde{M} \rangle_T}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\widetilde{M}_T}{\epsilon}. \end{aligned}$$

Note that

$$\begin{aligned} \widetilde{M}_T &= - \int_0^T \left(K_H^{-1} \int_0^\cdot \eta_r dr \right) (s) dW_s \\ &= - \frac{1}{\Gamma(\frac{1}{2} - H)} \int_0^T s^{H - \frac{1}{2}} \int_0^s \frac{r^{\frac{1}{2} - H}}{(s - r)^{H + \frac{1}{2}}} \eta_r dr dW_s \\ &= \frac{1}{\Gamma(\frac{1}{2} - H)} \int_0^T s^{H - \frac{1}{2}} \int_0^s \frac{r^{\frac{1}{2} - H}}{(s - r)^{H + \frac{1}{2}}} [b(r, X_r^\epsilon) - b(r, X_r)] dr dW_s \\ &\quad + \epsilon \frac{1}{\Gamma(\frac{1}{2} - H)} \int_0^T s^{H - \frac{1}{2}} \int_0^s \frac{r^{\frac{1}{2} - H}}{(s - r)^{H + \frac{1}{2}}} \frac{y}{T} dr dW_s, \end{aligned}$$

therefore by (H3), we conclude that

$$\lim_{\epsilon \rightarrow 0} \frac{R_\epsilon - 1}{\epsilon} = \frac{1}{\Gamma(\frac{1}{2} - H)T} \int_0^T s^{H-\frac{1}{2}} \left[\int_0^s \frac{r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} (1 + \partial_2 b(r, X_r)(T-r)) dr \right] y dW_s.$$

The proof is complete.

Remark 4.2 If $H = \frac{1}{2}$, i.e. B^H is a Brownian motion, then the corresponding derivative formula is of the following type

$$D_y P_T f(x) = \frac{1}{T} \int_0^T [1 + (T-s) \partial_2 b(s, X_s)] y dW_s.$$

Remark 4.3 Since we deal with one dimensional case, the derivative formula of theorem (4.1) is equivalent to $P_T f(\cdot)$ is derivative. The method we adopt is also valid for n -dimensional case.

As an application of the derivative formula derived above, we have the following result.

Corollary 4.4 If (H3) holds, then for any nonnegative $f \in \mathcal{B}_b(\mathbb{R})$ and $t > 0, x, y \in \mathbb{R}$,

$$(P_T f(y))^p \leq P_T f^p(x) \exp\left[\frac{p}{p-1} C(T, \overline{K}, H) |y-x|^2\right],$$

where $C(T, \overline{K}, H) = \left(\frac{B(\frac{3}{2}-H, \frac{1}{2}-H)}{\Gamma(\frac{1}{2}-H)}\right)^2 \frac{(1+\overline{K}T)^2}{T^{2H}2(1-H)}.$

Proof. By (4.1) and the Young inequality [6], we have. for all $\delta > 0$,

$$|D_y P_T f(x)| \leq \delta [P_T(f \log f)(x) - (P_T f)(x)(\log P_T f)(x)] + P_T f(x) [\delta \log \mathbb{E} e^{\frac{1}{\delta} N_T}]. \quad (4.3)$$

Now let $\beta(s) = 1 + s(p-1)$, $\gamma(s) = x + s(y-x)$, $s \in [0, 1]$, we have

$$\begin{aligned} & \frac{d}{ds} \log(P_T f^{\beta(s)})^{\frac{p}{\beta(s)}}(\gamma(s)) \\ &= \frac{p(p-1)}{\beta^2(s)} \frac{P_T(f^{\beta(s)} \log f^{\beta(s)}) - (P_T f^{\beta(s)}) \log P_T f^{\beta(s)}}{P_T f^{\beta(s)}}(\gamma(s)) + \frac{p}{\beta(s)} \frac{D_{y-x} P_T f^{\beta(s)}}{P_T f^{\beta(s)}}(\gamma(s)) \\ &\geq \frac{p}{\beta(s) P_T f^{\beta(s)}(\gamma(s))} \left\{ \frac{p-1}{\beta(s)} [P_T(f^{\beta(s)} \log f^{\beta(s)})(\gamma(s)) \right. \\ &\quad \left. - (P_T f^{\beta(s)}) \log P_T f^{\beta(s)}(\gamma(s))] - |D_{y-x} P_T f^{\beta(s)}|(\gamma(s)) \right\} \\ &\geq -\frac{p(p-1)}{\beta^2(s)} \log \mathbb{E} e^{\frac{1}{\delta} N_T}, \end{aligned}$$

where we use (4.3) and choose $\delta = \frac{p-1}{\beta(s)}$ for the last inequality, note that N_T is corresponding to the direction $y-x$.

Next we are to estimate $\mathbb{E} e^{\frac{1}{\delta} N_T}$. Since $\mathbb{E} e^{\frac{1}{\delta} N_T} \leq \left(\mathbb{E} e^{\frac{2}{\delta^2} \langle N_T \rangle} \right)^{\frac{1}{2}}$, we turn to the term $\langle N_T \rangle$.

$$\langle N_T \rangle = \frac{1}{(\Gamma(\frac{1}{2} - H)T)^2} \int_0^T s^{2H-1} \left[\int_0^s \frac{r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} (1 + \partial_2 b(r, X_r)(T-r)) dr \right]^2 (y-x)^2 ds$$

$$\begin{aligned}
&\leq \left(\frac{1 + \overline{K}T}{\Gamma(\frac{1}{2} - H)T} \right)^2 \int_0^T s^{2H-1} \left[\int_0^s \frac{r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} dr \right]^2 (y-x)^2 ds \\
&= \left(\frac{B(\frac{3}{2} - H, \frac{1}{2} - H)}{\Gamma(\frac{1}{2} - H)} \right)^2 \frac{(1 + \overline{K}T)^2}{T^{2H}2(1-H)} (y-x)^2 \\
&=: C(T, \overline{K}, H)(y-x)^2.
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
&\frac{d}{ds} \log(P_T f^{\beta(s)})^{\frac{p}{\beta(s)}}(\gamma(s)) \\
&\geq -\frac{p}{p-1} C(T, \overline{K}, H)(y-x)^2.
\end{aligned}$$

Integrating on the interval $[0, 1]$ with respect to s , we get the desired result.

Remark 4.5 To our knowledge, for the results of theorem 3.1 and corollary 4.4, we can not decide which is better.

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